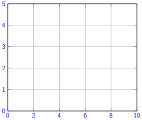
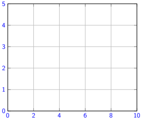
Chapter 4.3: Monotonic Functions and the First Derivative Test

Increasing/Decreasing

$f(x)$ is increasing on $[a, b]$ if all c, d with $a \leq c < d \leq b$ satisfy $f(c) < f(d)$.
(f is going up as we move from left to right.)



$f(x)$ is decreasing on $[a, b]$ if all c, d with $a \leq c < d \leq b$ satisfy $f(c) > f(d)$.
(f is going down as we move from left to right.)



If $f(x)$ is a continuous function for $a \leq x \leq b$ and it is differentiable for $a < x < b$, then we say

- $f(x)$ is increasing on $[a, b]$ if $f'(x) > 0$ for all $a < x < b$
- $f(x)$ is decreasing on $[a, b]$ if $f'(x) < 0$ for all $a < x < b$

$f'(x)$ can change sign if

- $f'(x) = 0$
- $f'(x)$ is not defined

If $f'(x)$ does not change sign on $[a, b]$, testing one point for sign is enough.

Functions which are either always increasing or always decreasing are called monotonic.

Finding where function is increasing and decreasing

Plan:

1. Find where derivative is 0 or undefined. (Critical points.)
2. Use these points to divide domain into intervals. Test a single point for each interval (only need the sign!).
3. Where you get a positive is an interval where function is increasing; where you get a negative is an interval where function is decreasing.

(If continuous, can combine consecutive intervals with same behavior.)

Example: Determine the intervals when the function $f(x) = x^3 - 3x^2 - 9x + 13$ is increasing and decreasing.

$f'(x) = 3x^2 - 6x^2 - 9$

Now solve $f'(x) = 0$

$$\begin{aligned} 3x^2 - 6x^2 - 9 &= 0 \\ x^2 - 2x^2 - 3 &= 0 \\ (x-3)(x+1) &= 0 \end{aligned}$$

Critical points are $-1, 3$. The intervals of interest are $(-\infty, -1], [-1, 3], [3, \infty]$

$f'(-2) = 5$ so f is increasing on $(-\infty, -1]$

$f'(0) = -3$ so f is decreasing on $[-1, 3]$

$f'(4) = 5$ so f is increasing on $[-3, \infty)$

Increasing on $(-\infty, -1]$ and $[-3, \infty)$

Decreasing on $[-1, 3]$

Example: Determine when the function $g(t) = t^5 + 5t^4 - 20t^3 - 31$ is increasing and when it is decreasing.

$g'(t) = 5t^4 + 20t^3 - 60t^2$
$t^4 + 4t^3 - 12t^2 = 0$
$t^2(t+6)(t-2) = 0$
critical points are $-6, 0, 2$.
$g(-10) = 100(-4)(-12) > 0$
$g(-1) = 1(5)(-3) < 0$
$g(1) = 1(7)(-1) < 0$
$g(3) = 9(9)(1) > 0$
g is increasing on $(\infty, -6]$ and $[2, \infty]$ g is decreasing on $[-6, 2]$

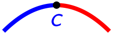
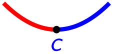
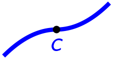
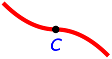
Example: Determine when the function $h(x) = e^{-4x}\left(x^{4/5} + x^{9/5}\right)$ is increasing and when it is decreasing.

$h'(x) = -4e^{-4x}\left(x^{4/5} + x^{9/5}\right) +$
$e^{-4x}\left(\frac{4}{5}x^{-1/4} + 9/5x^{4/5}\right)$
$h'(x) =$
$\frac{1}{5}e^{-4x}\left(-20x^{4/5} - 20x^{9/5} + 4x^{-1/5} + 9x^{4/}\right.$
$h'(x) =$
$\frac{e^{-4x}}{5}\left(-11x^{4/5} - 20x^{9/5} + 4x^{-1/5}\right)$
$h'(x) = \frac{e^{-4x}}{5}x^{-1/5}\left(-11x^1 - 20x^2 + 4\right)$
$h'(x) = \frac{e^{-4x}}{5}x^{-1/5}(-4x+1)(5x+4)$
Critical points: $-\frac{4}{5}, 0, \frac{1}{4}$
$x < -\frac{4}{5} : f(x) = + \cdot - \cdot + \cdot - > 0$
$-\frac{4}{5} < x < 0 : f(x) = + \cdot - \cdot + \cdot + < 0$
$0 < x < \frac{1}{4} : f(x) = + \cdot + \cdot + \cdot + > 0$
$\frac{1}{4} < x : f(x) = + \cdot + \cdot - \cdot + < 0$
increasing on $(-\infty, -\frac{4}{5}]$ and $[0, \frac{1}{4}]$
decreasing on $[-\frac{4}{5}, 0]$ and $[\frac{1}{4}, \infty)$

First Derivative Test

classification of critical points

The first derivative gives information about when function is increasing/decreasing. It can be used to determine if a critical point is a local max/min.

$f'>0$ $f'<0$ c		Local max
$f'<0$ $f'>0$ c		Local min
$f'>0$ $f'>0$ c		Neither
$f'<0$ $f'<0$ c		Neither

Example: Classify critical point for

- $f(x) = x^3 - 3x^2 - 9x + 13$
  Increasing on $(-\infty, -1] \cup [-3, \infty)$
  Decreasing on $[-1, 3]$
  -1 is a local max
  3 is a local min
- $g(t) = t^5 + 5t^4 - 20t^3 - 31$
  g is increasing on $(\infty, -6] \cup [2, \infty]$
  g is decreasing on $[-6, 2]$
  -6 is a local max
  0 is not a local optimum
  2 is a local min
- $h(x) = e^{-4x}\left(x^{4/5} + x^{9/5}\right)$
  increasing on $(-\infty, -\frac{4}{5}] \cup [0, \frac{1}{4}]$
  decreasing on $[-\frac{4}{5}, 0] \cup [\frac{1}{4}, \infty)$
  $-\frac{4}{5}$ is a local max
  0 is a local min
  $\frac{1}{4}$ is a local max

Example: Find and classify the critical points of the function $f(x) = \frac{y+1}{y^2+8}$.

$$f'(x) = \frac{(y^2+8) - 2y(y+1)}{(y^2+8)^2} = \frac{y^2+8-2y^2-2y}{(y^2+8)^2} = -\frac{y^2+2y-8}{(y^2+8)^2}$$

$$f'(x) = \frac{-(y+4)(y-2)}{(y^2+8)^2}$$

Critical points are -4 and $+2$.

$f'(-10) = -(-6)(-12)/(100+8)^2 < 0$
$f'(0) = -(4)(-2)/(0+8)^2 > 0$
$f'(10) = -(14)(8)/(100+8)^2 < 0$
f is decreasing on $(-\infty, -4] \cup [2, \infty)$

f is increasing on $[-4, 2]$

-4 is a local minimum
2 is a local maximum
TODO: Draw the lines with intervals and + and -.